colourful Mathematics 10

MOZAIK

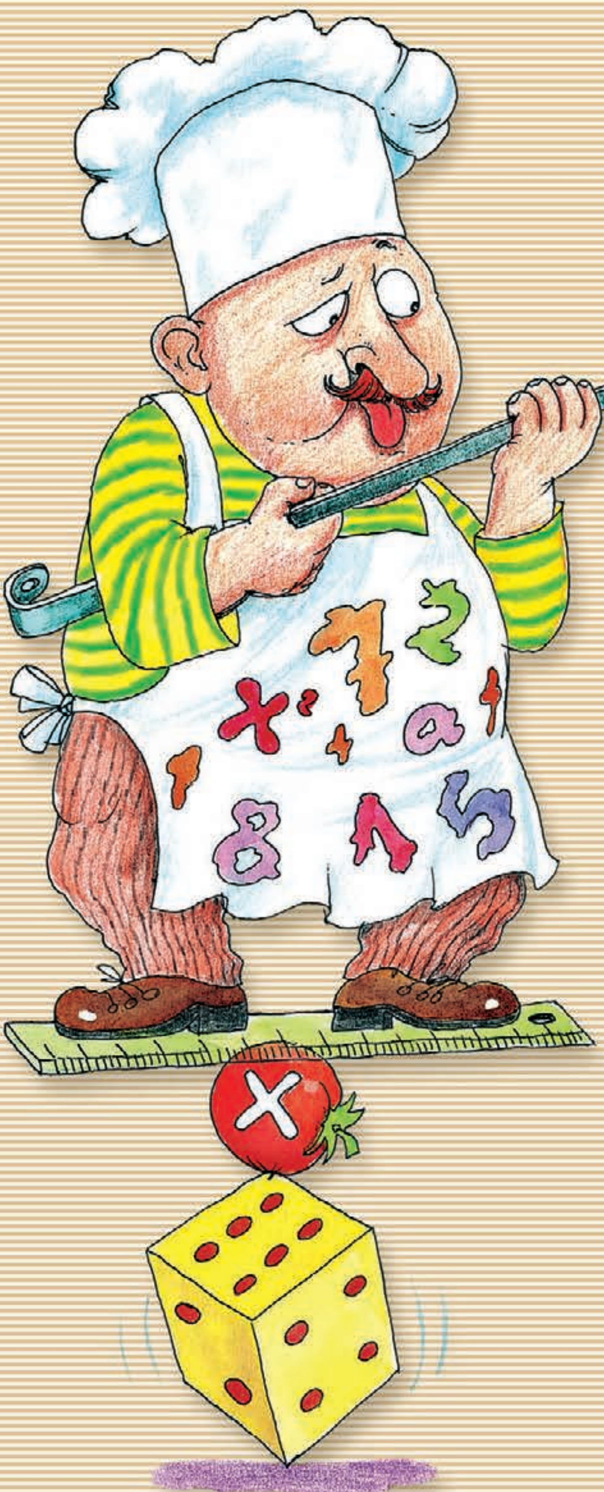
József Kosztolányi
István Kovács
Klára Pintér
János Urbán
István Vincze

Mathematics

textbook

10

Mozaik Education – Szeged, 2015



Mathematical reasoning 1

Computing the root 2

The quadratic equation 3

Geometry 4

Trigonometric functions 5

Calculation of probability 6

TABLE OF CONTENTS

Mathematical reasoning

Computing the root

The quadratic equation

Geometry

Trigonometric functions

Calculation of probability

Guide to use the course book

The notations and highlights used in the book help with acquiring the courseware.

- The train of thought of the worked examples show samples how to understand the methods and processes and how to solve the subsequent exercises.
- The most important definitions and theorems are denoted by colourful highlights.
- The parts of the courseware in small print and the worked examples noted in claret colour help with deeper understanding of the courseware. These pieces of knowledge are necessary for the higher level of graduation.
- Figures, the key points of the given lesson, review and explanatory parts along with interesting facts of the history of mathematics can be found on the margin.

The difficulty level of the examples and the appointed exercises is denoted by three different colours:

Yellow: drilling exercises with basic level difficulty; the solution and drilling of these exercises is essential for the progress.
Blue: exercises the difficulty of which corresponds to the intermediate level of graduation.
Claret: problems and exercises that help with preparing for the higher level of graduation.

These colour codes correspond to the notations used in the Colourful mathematics workbooks of Mozaik Education. The workbook series contains more than 3000 exercises which are suitable for drilling, working on in lessons and which help with preparing for the graduation.

The end results of the appointed exercises can be found on the following website: *www.mozaik.info.hu*. Website *www.mozaweb.com* offers more help material for processing with the course book.

◆ ◆ ◆

Mathematics, "ratio" and logic way of thinking are probably the most efficient tools of cognition of our world, which sometimes associate with unexplainable phenomena. These are inseparable from Homo sapiens and these make the everyday activities complete.

A few thoughts from those who have experienced all this:

"So what is ratio the human intellect created logic out of? It is obvious that it is 'there in' the nature, otherwise it would not be possible to understand the nature with the help of rational tools. Ratio unites humans, animals and nature." (Imre Kertész, Nobel prize-winning Hungarian writer)

"Through the centuries the collective awareness of mathematicians created its own universe. Where it is I do not know – and I think the word "where" also loses its meaning here –, but I can assure the reader: this mathematical universe is all too real to those who live in it. The mankind could pierce into the mystery of the surrounding world the most deeply right by means of Mathematics." (Ian Stewart)

"The strict proof is usually the last step! Before that many conjectures are needed, and for these aesthetic belief is extremely important." (Roger Penrose)

The Authors wish productive work and learning.

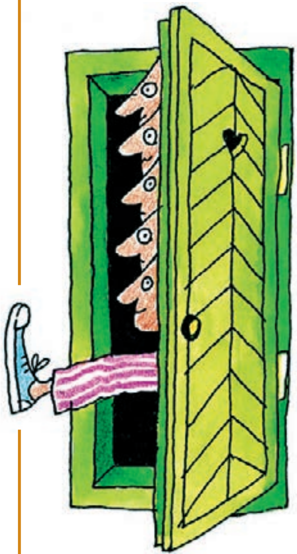
Mathematical reasoning

René Descartes (1569–1650) was looking for a general method to solve problems in his work "Regulae ad directionem ingenii" (Rules for the Direction of the Mind). His idea was firstly to express every problem as a mathematical problem, then secondly to express every mathematical problem as an algebraic one, and then finally to solve these in the form of equations.

For the development of our way of thinking it is also necessary to observe the structure, the schemes of the reflective action. Henceforth we are going to apply a few methods which give a useful strategy for solving many problems. The more strategies we know, the greater our chance to be able to solve the problems facing us.

3. Arrangement (ordering) problems

Arrangement (ordering) of distinct objects



Example 1

Andrew, Bruce, Chris, Denis and Evan arrive to a party. They are waiting in front of the door to decide in which order they should enter. In how many different orders can they enter the room if one person can step in at once?

Solution

We can select the first person to enter in 5 different ways.

Since four people entering as second can be selected to each person entering as first, the person entering as second can be selected from only the 4 people still standing in front of the door, i.e. in 4 different ways, thus the people entering as first and as second can be selected in $5 \cdot 4$ different ways.

Since the person entering as third can be selected from only the 3 people still standing in front of the door, i.e. in 3 different ways, thus the people entering as first, second and third can be selected in $5 \cdot 4 \cdot 3$ different ways.

Since the person entering as fourth can be selected from only the 2 people still standing in front of the door, i.e. in 2 different ways, thus the people entering as first, second, third and fourth can be selected in $5 \cdot 4 \cdot 3 \cdot 2$ different ways.

The person entering as fifth can only be the 1 person still standing in front of the door, thus all the possible entering orders of the five people are: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Example 2

Andrew, Bruce, Chris, Denis and Evan sit down at a round table in a party. We only concentrate on the neighbours to the right and to the left. For example the arrangements in figure 20 are considered the same. There is no special place, e.g. for an honoured guest, at the table. In how many different ways can the guests sit down?

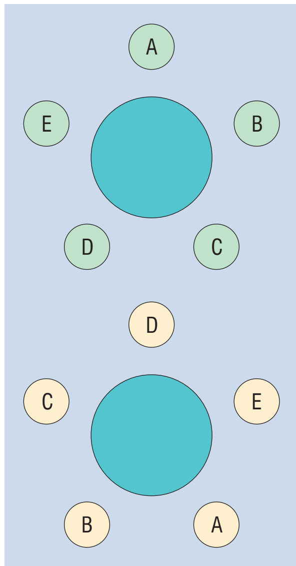


Figure 20

Solution I

We make Andrew sit. This way we have determined his place at the "rotatable" table (after this the table cannot be rotated anymore, as if it were a straight table). Thus we can make the remaining 4 people sit in $4 \cdot 3 \cdot 2 \cdot 1$ different ways. Therefore 5 people can sit down at a round table in $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different ways.

Solution II

If we are thinking so that the first person can be seated in 5 different ways, then the others can be seated in $4 \cdot 3 \cdot 2 \cdot 1$ different ways; there would be a total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ possibilities.

However among those arrangements, where everyone sits to the place to his neighbour to the right, give the same order of seating. They can do this 5 times in a row before they get back to their initial places, i.e. every seating arrangement appears 5 times among the $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ possibilities.

Thus the number of distinct seating arrangements is:

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5} = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

The number of arrangements, if there are like objects

Example 3

How many five-letter “words” can be made from the set of letters a, a, a, m and m?

Solution

If we write the like letters in different forms, for example A, *a*, a, M, m, then we are looking for all the possible arrangements of 5 distinct signs which, based on the above, is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Since these three types of the letter “a” represent the same letter in the word, the cases, which differ in the different orders of the distinct letters “a”, are not considered as different cases. The letters “a” can be exchanged in $3 \cdot 2 \cdot 1$ different ways. Regarding the arrangement of the letters “a” these $3 \cdot 2 \cdot 1 = 6$ cases count as 1, if the letters “a” are alike. Thus only $\frac{1}{6}$ of the so far counted cases mean actually distinct cases with respect to the letters “a”.

The same way the different cases of the arrangements of the letters “m” are not distinct either, they can be exchanged in $2 \cdot 1$ different ways. Thus only $\frac{1}{2 \cdot 1}$ of the so far counted cases mean actually distinct cases with respect to the letters “m”.

Let us also count the possibilities with the help of the graph representation!

Therefore the number of five-letter words made from the letters a, a, a, m and m is:

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = 10.$$

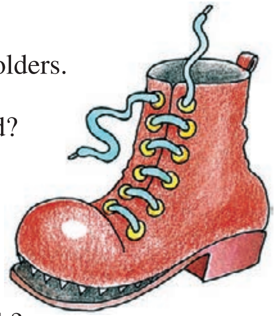
Exercises

1. 4 married couples arrive at a party.
   a) In how many different orders can they enter the room if they arrive at once?
   b) In how many different orders can they enter the room if every husband steps in right after his wife?
   c) In how many different ways can they sit down at a round table?
   d) In how many different ways can they sit down at a round table if every husband sits next to his wife?

2. How many different tunes with 7 notes can be produced by the computer program which plays notes C, C, G, G, G, A, A in all the possible orders? (All the notes are of equal length.)

3. Which one is greater: the number of the seven-letter words that can be created from the letter set of (a, a, a, a, b, b, b) or the number of the six-letter words that can be created from this same letter set?

4. There are 3 green and 3 blue holders in a set of soft-boiled egg holders. These can be strung on a rod either with the bottom up or down. In how many different orders can these holders be strung on a rod?



5. A boot can be laced up in many ways. A boot and a lacing can be seen in the figure, but the bootlace can be laced inside the boot in several ways. In how many different ways?

6. Four cars each have a number plate removed.

a) In how many different ways can the number plates be put back?

b) In how many different ways can the four number plates be put back so that exactly one will be in the correct place?

c) In how many different ways can the four number plates be put back so that exactly three will be in the correct place?

7. A secretary wrote 5 letters and also addressed 5 envelopes for those who the letters were for, but she did not have time to put the letters in the envelopes. The office assistant put the letters in the envelopes at random.

a) In how many different ways could he put the letters in the envelopes?

b) In how many different ways can the letters be put in the envelopes so that exactly one letter will be in the correct place?

c) In how many different ways can the letters be put in the envelopes so that exactly three letters will be in the correct place?

8. How many necklaces can we string using 3 white and 5 red beads if same coloured beads are alike?

Andrew's solution: First we plan the places of the white beads, then we put the red ones on the three arcs defined by the white ones.

There are 5 possibilities:

$0+0+5$; $0+1+4$; $0+2+3$; $1+1+3$; $1+2+2$.

Therefore 5 different necklaces can be strung.

Bruce's solution: 8 different beads can be strung on a round necklace in $7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$ different ways. Now there are 5 red like beads and 3 white like beads, so it should be divided by $5\cdot4\cdot3\cdot2\cdot1$ and also by $3\cdot2\cdot1$, thus the result will be 7 possibilities.

Therefore 7 different necklaces can be strung.

Whose solution is correct?

9. In how many different orders can 5 basketball players with different heights enter the court if they enter one after the other and none of them can go between two taller players?

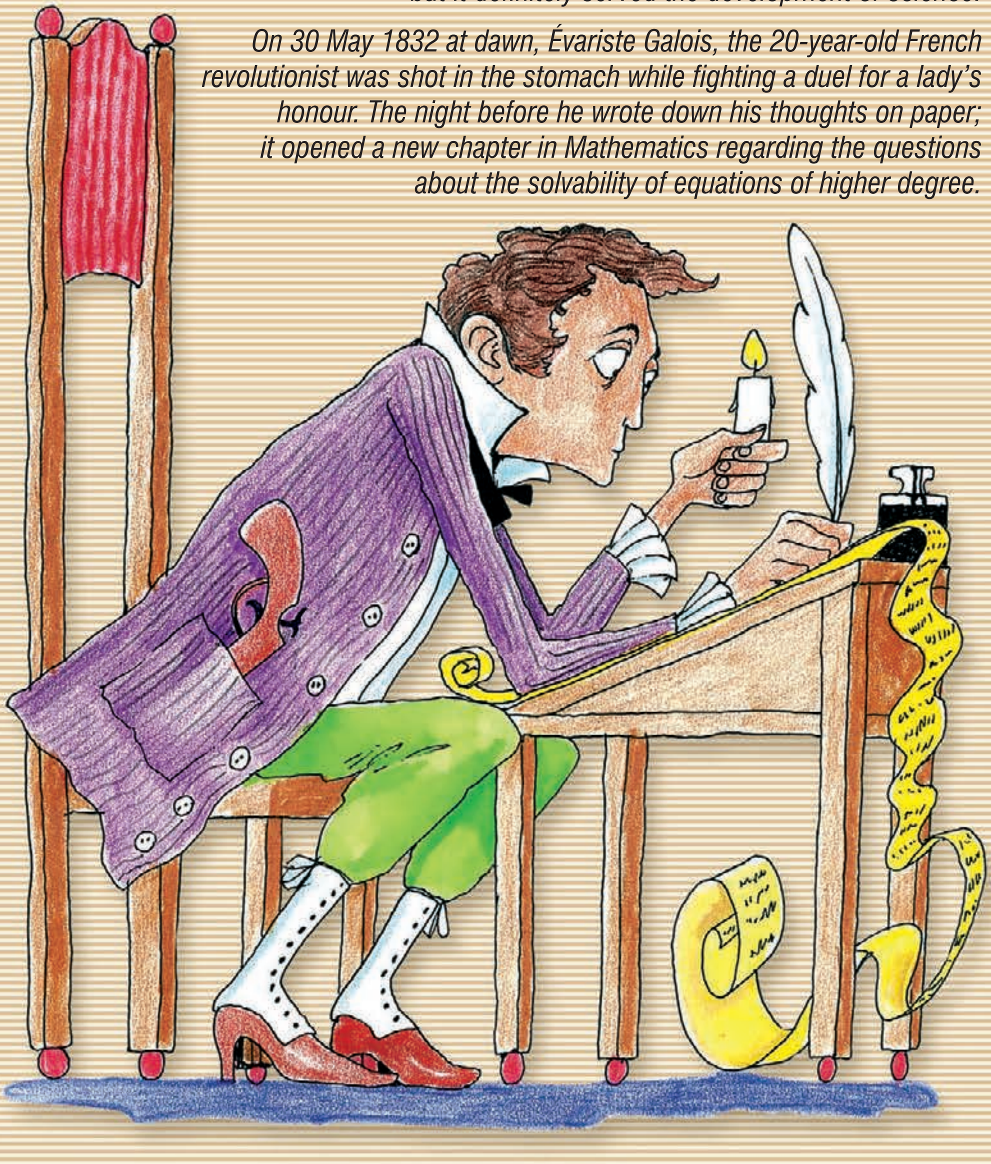
The quadratic equation

The chapter of the history of Mathematics dealing with equations is full of moments which could be the story of a novel. The daring and visionary thoughts were not rewarded by the contemporaries in every case.

The Greek Hippasus is said to have drowned in the sea as a punishment for his lateral way of thinking.

The Mathematicians of the 16th century were constantly competing. It sometimes meant glory, other times it meant shame for the rivals, but it definitely served the development of science.

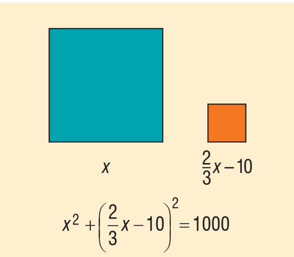
On 30 May 1832 at dawn, Évariste Galois, the 20-year-old French revolutionist was shot in the stomach while fighting a duel for a lady's honour. The night before he wrote down his thoughts on paper; it opened a new chapter in Mathematics regarding the questions about the solvability of equations of higher degree.



1. The quadratic equation and function

Several exercises and practical problems are easy to solve by applying equations. However it often happens that letters denoting the unknowns are raised to a power of higher degree in the equation set up. In this chapter we are looking for processes with the help of which we can determine the solutions of even these equations.

Based on the cuneiform script clay tables burnt in Mesopotamia around 2000 BC it can be established that people could solve these types of exercises confidently at that time. The following exercise originates from that time:



The sum of the area of two squares is 1000 area units. The side of one of the squares is 10 less than the 2/3 of the side of the other square. How long is a side of the square?

When denoting the side of the square mentioned in the exercise by x, the following equation results:

$$x^2+\left(\frac{2}{3}x-10\right)^2=1000,$$

$$x^2+\frac{4}{9}x^2-\frac{40}{3}x+100=1000,$$

$$9x^2+4x^2-120x+900=9000,$$

$$13x^2-120x-8100=0.$$

◆ ◆ ◆

We know the formation of one of the famous mathematical exercises of the Greek golden age thanks to *Plutarchus,* a Greek essayist. The people of the island of Delos were suffering from a plague and therefore they consulted the Delphi Oracle. They got the answer that, if they wanted to remove the plague, then they should double the size of the alter of Apollo's temple, which itself was a cube. This is where the exercise known as the *Delian problem* originates from:

The side of the cube is to be constructed the volume of which is double of the volume of a given cube.

They could not solve the exercise. The discovery of incommensurable ratios is credited to Hippasus of Metapontum. The Pythagoreans, who propagated that every phenomenon of the universe originated in whole numbers or in their ratios, made Hippasus drown in the sea as a punishment.

If we choose the volume of the original alter stone to be the unit, then the following is true for side x of the cube in question:

$$x^3=2.$$

◆ ◆ ◆

The signals, which would prove the existence of extra-terrestrial life and intelligence, are tried to be filtered out of the flood of information coming from the Universe with the help of huge aerial arrays. At the same time much information is radiated from the Earth into every direction of the space. The tool built for sending and receiving these data is called the *dish aerial*. The surface of fluid in a rotating glass forms the same surface.

We have already studied about the properties of the quadratic function. In the below example we brush up on this knowledge.

Example 1

Let us plot the graph of and characterise the following function:

$$f(x) = 2x^2 + 12x + 16.$$

Solution

Before drawing the graph we rewrite the given assignment rule to a form from which it is easy to see the necessary transformations. This process is called **completing the square.**

$$f(x) = 2x^2 + 12x + 16 = 2(x^2 + 6x) + 16 = 2(x^2 + 6x + 9 - 9) + 16 =$$
$$= 2[(x + 3)^2 - 9] + 16 = 2(x + 3)^2 - 18 + 16 = 2(x + 3)^2 - 2.$$

We established the following properties of the quadratic function $f_1(x) = x^2$ ***(Figure 1)***:

(1) It is an even function.

(2) Its zero is at $x = 0$.

(3) If $x \le 0$, then the function is strictly monotonically decreasing, and if $x \ge 0$, then it is strictly monotonically increasing.

(4) At $x = 0$ the function has a minimum, and its value is $f_1(0) = 0$.

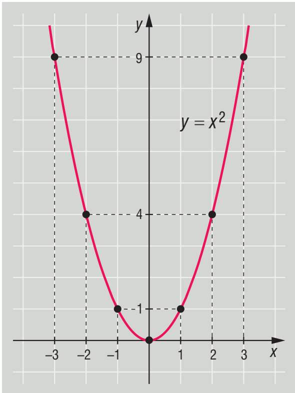


Figure 1

We can get the graph of function f in question from the standard parabola of the graph of the function $f_1(x) = x^2$ so that we transform it based on the assignment rule resulting by completing the square. The steps are as follows:

$f_2(x) = (x + 3)^2$: translation along the x-axis in the negative direction, the vertex of the parabola is $P(-3; 0)$.

$f_3(x) = 2(x + 3)^2$: we double the values of the function. A stretched parabola is resulting, every point of which is at twice the distance from the x-axis.

$f(x) = 2(x + 3)^2 - 2$: translation along the y-axis by 2 units in the negative direction, thus the vertex will be at $C(-3; -2)$ ***(Figure 2)***.

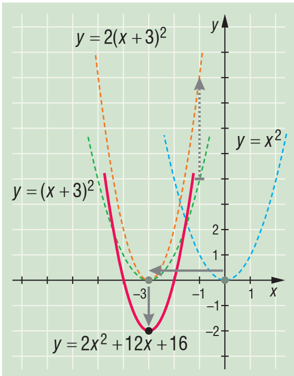


Figure 2

A few **properties** of the function:

(1) It has two zeroes: $x_1 = -4$ and $x_2 = -2$.

(2) If $x \le -3$, then the function is strictly monotonically decreasing, and if $x \ge -3$, then it is strictly monotonically increasing.

(3) At $x = -3$ the function has a minimum, and its value is $f(-3) = -2$.

According to the above we can in general state the following:

The assignment rule of the quadratic function in general is as follows:

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = ax^2 + bx + c, \text{ where } a \in \mathbb{R} \setminus \{0\};\ b, c \in \mathbb{R}.$$

If $a > 0$, then the graph is a parabola open upwards.

If $a < 0$, then it is a parabola open downwards.

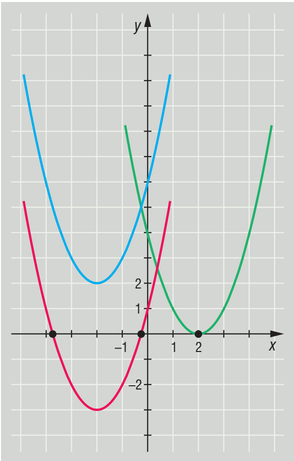


Figure 3

It is practical to rewrite the assignment rule of the quadratic functions into the following *completed square* form:

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = a(x - u)^2 + v, \text{ where } a \in \mathbb{R} \setminus \{0\};\ u, v \in \mathbb{R}.$$

The standard parabola is stretched by a factor of a, and is translated by the vector $\vec{v}(u; v)$ so that the vertex of the parabola will be at the point $C(u; v)$.

A real number x, for which $f(x) = 0$, is called a *zero* of the function.

A quadratic function can have 0, 1 or 2 zeroes, since the parabola, depending on its position, can intersect the x-axis at two points at most. *(Figure 3)*

Example 2

Let us determine the value of q in the following function so that the substitution value of the function will be positive for every real number x.

$$f(x) = x^2 + 2qx + q^2 + 2q - 4.$$

Solution

Since the coefficient of x^2 is positive, the parabola is open upwards, thus it is possible that it takes a positive value for every real number.

At first let us transform the assignment rule of the function by completing the square:

$$f(x) = x^2 + 2qx + q^2 + 2q - 4 = (x + q)^2 + 2q - 4.$$

Since the first term of this expression is a quadratic one, and it is non-negative in every case, the substitution value of the function will be positive in every case, if the following inequality is satisfied:

$$2q - 4 > 0,$$
$$q > 2.$$

So this is how we have to choose the value of q because of the given conditions. Then the graph of the function is translated along the y-axis in the positive direction.

Exercises

1. Complete the square in the following expressions.

a) $x^2 - 4x + 4;$ *b)* $x^2 - 6x + 8;$ *c)* $x^2 + 8x - 2;$

d) $2x^2 + 8x - 5;$ *e)* $-x^2 + 8x - 2;$ *f)* $-3x^2 + 6x + 1.$

2. Plot the graphs of the following functions defined on the set of real numbers in one coordinate system.

a) $f_1(x) = x^2,\ \ f_2(x) = 2x^2,\ \ f_3(x) = 2(x-1)^2,\ \ f_4(x) = 2(x-1)^2 + 1;$

b) $f_1(x) = x^2,\ \ f_2(x) = \frac{1}{2}\cdot x^2,\ \ f_3(x) = \frac{1}{2}\cdot(x+1)^2,\ \ f_4(x) = \frac{1}{2}\cdot(x+1)^2 - 2;$

c) $f_1(x) = x^2,\ \ f_2(x) = -3x^2,\ \ f_3(x) = -3(x-1)^2,\ \ f_4(x) = -3(x-1)^2 - 1.$

3. Plot the graphs of and characterise the following functions defined on the set of real numbers.

a) $f(x) = x^2 - 4x + 3;$ *b)* $f(x) = -x^2 - 4x + 3;$ *c)* $f(x) = 2x^2 - 4x + 3.$

Determine the zeroes of the functions, if they exist, and the maximum or minimum places and values.

4. In the following functions defined on the set of real numbers determine the value of q so that the function will have 0; 1; 2 zeroes.

a) $f(x) = x^2 - 4x + q;$ *b)* $f(x) = -x^2 - 4x + q;$ *c)* $f(x) = 2x^2 - 4x + q.$

5. In the function $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + px + q$ determine the values of p and q so that the function

a) takes its minimum at the place $x = 1$, and its value is 2;

b) takes its minimum at the place $x = -1$, and its value is -2;

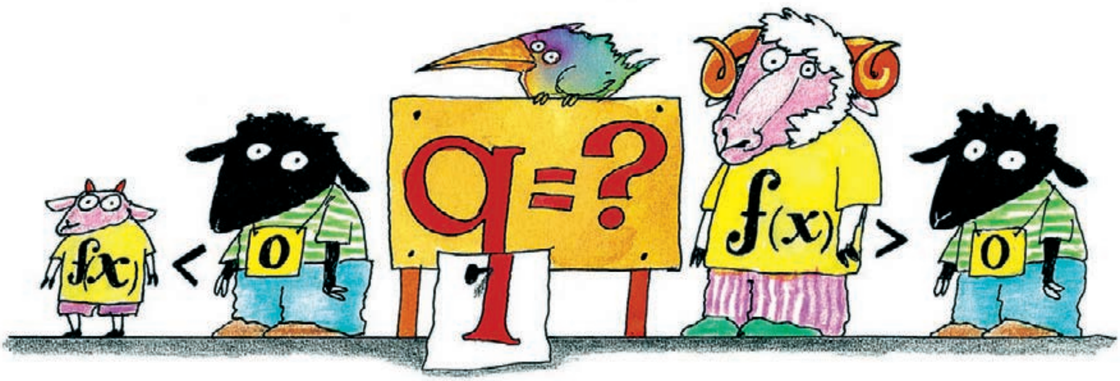
c) takes its minimum at the place $x = 4$, and its value is -3.

6. How shall we choose the value of q so that every value of the following functions is positive on the set of real numbers?

a) $f(x) = x^2 + 6x + q;$ *b)* $f(x) = x^2 - 4x + q;$ *c)* $f(x) = 2x^2 + 8x + q.$

7. How shall we choose the value of q so that every value of the following functions is negative on the set of real numbers?

a) $f(x) = -x^2 - 4x + q;$ *b)* $f(x) = -x^2 + 2x + q;$ *c)* $f(x) = -2x^2 - 4x + q.$

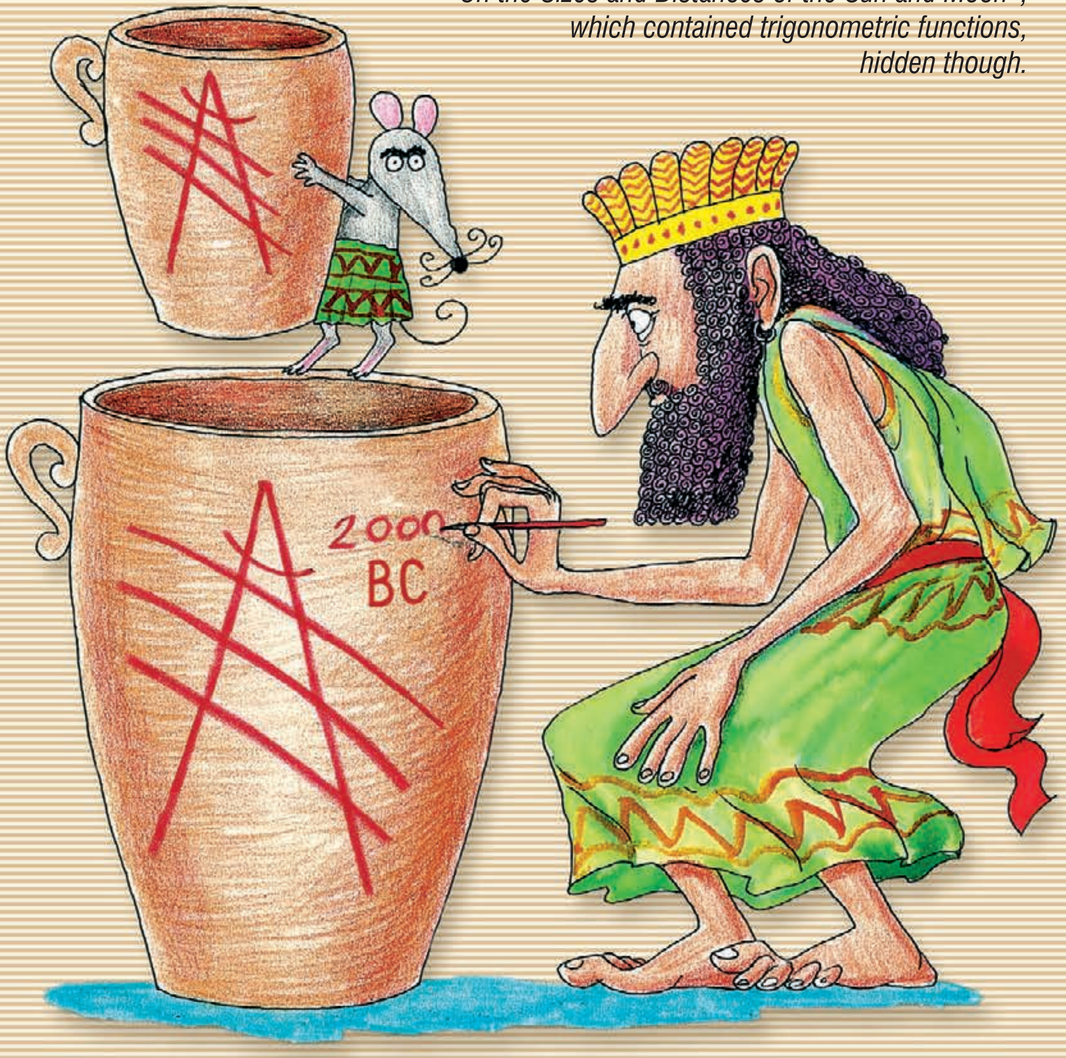


Geometry

Based on the clay pots, which were found in 1936 in the course of diggings around Susa, located in today's Iran, it seems so that the Babylonians used the intercept theorem and the concept of similarity about 4000 years ago.

In the 5th century BC Hippocrates of Chios used the relation between the inscribed and the central angles of the circle, and the similarity in his proofs, and could construct the geometric mean of two line segments.

In the 3rd century BC Aristarchus of Samos performed such calculations and approximations in his only surviving work with the title "On the Sizes and Distances of the Sun and Moon", which contained trigonometric functions, hidden though.



2. The theorem of the central and inscribed/tangent-chord angles

Definition: **If the vertex of a convex angle is on the circumference of a given circle, and the two arms of the angle are on two chords of the circle, then the angle is called an *inscribed angle* of the circle.** ***(Figure 2)***

Definition: **If the vertex of a convex angle is on the circumference of a given circle, and one of the arms of the angle is on one chord of the circle and the other arm is a tangent line of the circle, then the angle is called a *tangent-chord angle* of the circle.** ***(Figure 3)***

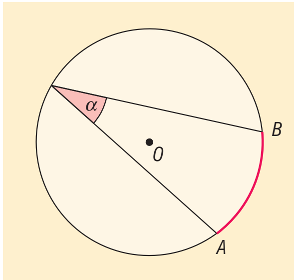


Figure 2

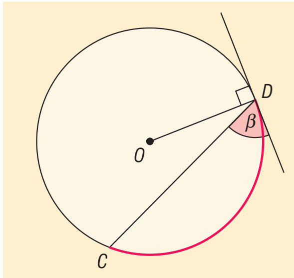


Figure 3

The part of the boundary line of the circle, which is inside the angular domain, is the arc intercepted by or corresponding to the given inscribed angle.

The following are also typical phrasings: the inscribed angle corresponding to a given arc, the inscribed angle subtended by a given arc. In figure 2 the angle α is an inscribed angle corresponding to the arc AB, in figure 3 the angle β is a *tangent-chord angle* corresponding to the arc CD.

We can make an interesting observation when we recall the Thales' theorem ***(Figure 4)***. The central angle corresponding to the arc AB (which in this case is a semicircle) is 180º, and the inscribed angles corresponding to this same arc are 90º, i.e. they are equal to the half of the central angle.

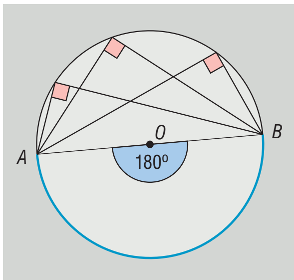


Figure 4

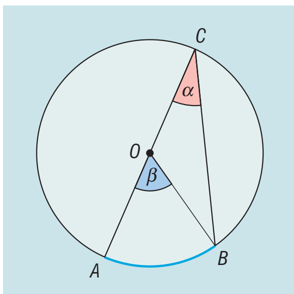
Now we are going to prove that it is also true in general, i.e. the following theorem holds:

Theorem: **The measure of any inscribed angle or any tangent-chord angle corresponding to a given arc in a given circle is half of the measure of the central angle corresponding to the same arc.**

Proof

One central angle and infinitely many inscribed angles and tangent-chord angles correspond to a given arc in a given circle; therefore we have to consider the different mutual positions of the inscribed angles/tangent-chord angles and the central angle. The proof has more steps to match these positions.

Figure 5



(1) The centre of the circle lies on one of the arms of the inscribed angle. ***(Figure 5)***

The angle AOB is an exterior angle of the isosceles triangle COB, therefore

$$\beta = AOB\sphericalangle = OBC\sphericalangle + BCO\sphericalangle = 2\alpha.$$

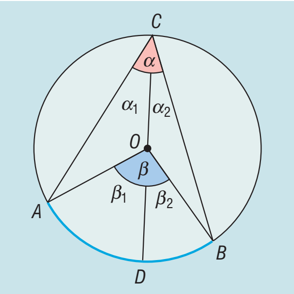


Figure 6

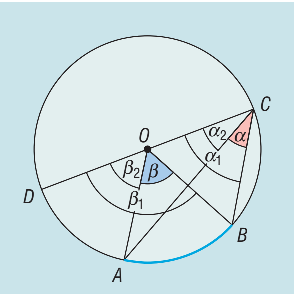


Figure 7

(2) The centre of the circle is an interior point of the angular domain of the inscribed angle. ***(Figure 6)***

Let us draw the diameter of the circle passing through the point C, and let its other end-point opposite C be D. The position of the inscribed and central angles corresponding to the arcs AD and DB matches the previous cases, thus

$$\beta_1 = AOD\sphericalangle = 2\cdot ACD\sphericalangle = 2\alpha_1, \quad \beta_2 = DOB\sphericalangle = 2\cdot DCB\sphericalangle = 2\alpha_2,$$

which implies

$$\beta = \beta_1 + \beta_2 = 2\alpha_1 + 2\alpha_2 = 2(\alpha_1 + \alpha_2) = 2\alpha.$$

(3) The centre of the circle is a point outside the angular domain of the inscribed angle. ***(Figure 7)***

Let us draw the diameter CD again. The position of the inscribed and central angles corresponding to the arcs DA and DB matches the ones in case (1); therefore

$$\beta_1 = DOB\sphericalangle = 2\cdot DCB\sphericalangle = 2\alpha_1, \quad \beta_2 = DOA\sphericalangle = 2\cdot DCA\sphericalangle = 2\alpha_2,$$

thus

$$\beta = \beta_1 - \beta_2 = 2\alpha_1 - 2\alpha_2 = 2(\alpha_1 - \alpha_2) = 2\alpha.$$

In the first three steps we have shown the theorem for all possible positions of inscribed angles. In the next step we are going to prove it for the tangent-chord angles.

(4) Let α denote the tangent-chord angle subtended by the arc AB. ***(Figure 8)***

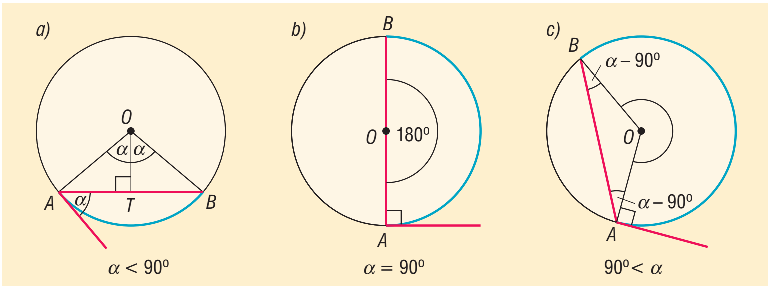


Figure 8

a) If $\alpha < 90^o$, and OT is the altitude belonging to the base of the isosceles triangle AOB, then the tangent-chord angle with vertex A and the angle AOT are angles with mutually perpendicular arms, thus $AOT\sphericalangle = \alpha$. Since OT bisects the angle AOB, therefore $AOB\sphericalangle = 2\alpha$.

b) If α is a right angle, then the statement is a direct consequence of the fact that the radius drawn to the point of tangency is perpendicular to the tangent line.

c) If α is an obtuse angle, then the measure of the angles on the base of the isosceles triangle AOB is $\alpha - 90^o$, thus the angle included between the legs is

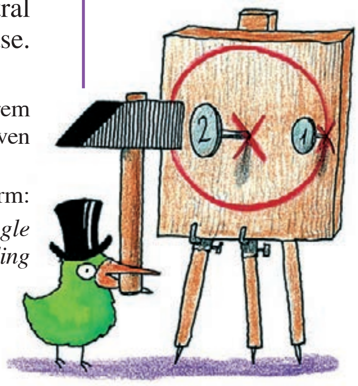
$$AOB\sphericalangle = 180^o - 2(\alpha - 90^o) = 360^o - 2\alpha,$$

which implies that the measure of the central angle corresponding to the highlighted arc AB is 2α.

Now the proof is finished: we have shown the theorem of the central and the inscribed angles/tangent-chord angles for each and every case.

Notes:

1. Thales' theorem (the central angle is 180º) is a special case of the theorem just proven; therefore by proving this more general theorem we have proven Thales' theorem too.
2. This theorem can also be rephrased in the following more general form: *In circles with equal radii the ratio of the measure of the central angle and the measure of the inscribed angle/tangent-chord angle corresponding to arcs with equal lengths is 2 : 1.*
   The corresponding arcs can be transformed to each other with the help of congruent transformations, thus the proof of this more general theorem originates in the proof of the original theorem.



Example 1

Three points of a circle divide it into three arcs the ratio of the length of which is 5 : 6 : 7. Find the measure of the central angles and the inscribed angles corresponding to the arcs.

Solution

In a given circle the central angle and the arc corresponding to it are directly proportional, thus if α, β, γ denote the corresponding central angles, then $\alpha : \beta : \gamma = 5 : 6 : 7$. Hence based on $\alpha + \beta + \gamma = 360º$:

$$\alpha = \frac{5}{18} \cdot 360º = 100º, \quad \beta = \frac{6}{18} \cdot 360º = 120º, \quad \gamma = \frac{7}{18} \cdot 360º = 140º.$$

Based on the theorem of the central and inscribed/tangent-chord angles the measures of the corresponding inscribed angles are:

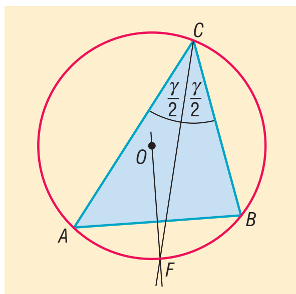
$$\frac{\alpha}{2} = 50º, \quad \frac{\beta}{2} = 60º, \quad \frac{\gamma}{2} = 70º.$$

It can be realised that the measure of the inscribed angles and the length of the corresponding arcs are directly proportional too.

Example 2

A triangle is given. Let us construct its circumscribed circle, the bisector of one of the interior angles and the perpendicular bisector of the side opposite this angle. What do you experience? Justify your observation. (Let us assume that two sides with distinct length meet at the vertex lying on the interior angle bisector.)

Figure 9



Solution

If our construction is precise enough, then we can find that the angle bisector and the perpendicular bisector intersect each other on the circumscribed circle. ***(Figure 9)***

Based on the figure we can also phrase the assumption that the intersection point is the midpoint of the arc AB.

We are going to show that both the angle bisector and the perpendicular bisector pass through the midpoint F of the arc AB.

Since the measure of the inscribed angle and the length of the arc corresponding to it are directly proportional, the angle bisector dividing the angle ACB into two equal parts also divides the arc AB into two equal parts, i.e. it passes through the midpoint of the arc.

All points of the perpendicular bisector of the side AB are equidistant from the points A and B, thus its intersection point with the arc AB can only be the midpoint of the arc.

With this we have justified our observation.

Exercises

1. What is the measure of the inscribed angles subtended by the arcs corresponding to the following central angles in a given circle?

a) 60º; *b)* 138º; *c)* $\frac{2\pi}{3}$; *d)* $\frac{5\pi}{4}$; *e)* 336º; *f)* β?

2. What is the measure of the central angle subtended by the arcs corresponding to the following inscribed angles in a given circle?

a) 15º; *b)* 84º; *c)* $\frac{3\pi}{4}$; *d)* $\frac{5\pi}{12}$; *e)* 216º; *f)* α?

3. The sum of the measure of the inscribed angle and the measure of the central angle corresponding to a given arc of a circle is

a) 180º; *b)* 210º; *c)* $\frac{5\pi}{2}$; *d)* $\frac{7\pi}{5}$; *e)* 411º; *f)* ω.

Calculate the measure of the inscribed angle and the central angle in each case.

4. Give the measure of the inscribed angle corresponding to the arc of the circle the length of which is

a) one third; *b)* $\frac{7}{12}$th; *c)* 60%; *d)* $\frac{11}{18}$th; *e)* $\frac{m}{n}$th

part of the circumference of the circle?

5. The vertices of a triangle divide the circumscribed circle into three arcs the ratio of the length of which is

a) $1:2:3$; *b)* $3:4:5$; *c)* $3:5:10$; *d)* $7:9:20$; *e)* $p:q:r$.

Calculate the measure of the interior angles of the triangle in each case.

6. Calculate the distance of the centre of a circle with a radius of 10 cm and the chord connecting the two end-points of an arc corresponding to a 120º central angle.

7. The angle included between the chord AB and the diameter AC in a circle with a radius of 10 cm is 30º. Calculate the length of the line segment BC.

8. Give the measure of the inscribed angle for which the following are true: one of its arms coincides the diameter of the circle and the other arm coincides a chord the length of which is equal to the radius of the circle.

VECTORS

1. The concept of a vector; the sum and the difference of vectors; scalar multiplication of vectors (reminder)

The concept of a vector

We got familiar with the concept of a vector, the definition of the sum and the difference of vectors, and the definition of the real scalar multiplication of vectors in year 9 in connection with translation. We are going to brush up this knowledge, sometimes by throwing different light upon it.

Definition: **If we distinguish the two end-points of a line segment in a way so that one of them is the *starting point* and the other one is the *end-point,* then a *directed line segment* results.**

Figure 92

In figure 92 the directed line segment pointing from A to B can be seen, with A as the starting point and B as the end-point.

Definition: **A directed line segment unambiguously defines a *vector.***

(1) and (2) and (3) $\Leftrightarrow$ (4)

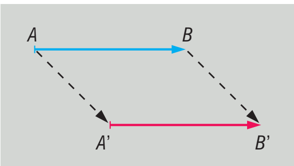


Figure 93

If the following are satisfied for the directed line segments pointing from A to B and pointing from A' to B' *(Figure 93)*:

(1) $AB = A'B'$, (2) the straight lines AB and $A'B'$ are parallel and (3) the line segments AA' and BB' do not have an interior point in common;

or

(4) there is a parallel translation, which takes A to A' and B to B' simultaneously,

then the two directed line segments **define the same vector.**

So for a given vector there are infinitely many directed line segments which define it. Any of these can be chosen for the representation of the vector. Vectors are represented by directed line segments graphically.

The notation of the vector defined by the directed line segment pointing from A to B: $\overrightarrow{AB}$.

Vectors are usually denoted by also a single lowercase letter (e.g. $\vec{a}$, $\vec{b}$), or in printed text by a “bold” lowercase letter (e.g. **a**, **b**).

Definition: **The length of the directed line segment defining the vector is the *absolute value or magnitude of the vector.***

Notation: $|\overrightarrow{AB}|$, $|\vec{a}|$, $|\mathbf{a}|$.

Definition: ***Two vectors are parallel*** **if the straight lines of the directed line segments defining them are parallel.**

Definition: **The vectors $\vec{a}$ and $\vec{b}$ *are unidirectional,* if they are parallel and pointing into the same direction, i.e. when drawing directed line segments representing $\vec{a}$ and $\vec{b}$ from a common starting point, then these are collinear, and their end-points coincide the same ray defined by the common starting point.** *(Figure 94)*

Definition: ***Two vectors are in the opposite directions,*** **if they are parallel but not unidirectional.** *(Figure 95)*

Definition: **If two vectors have the same magnitude (absolute value) and are in the opposite directions, then the *two vectors are each other's negative.*** *(Figure 96)*

Notation: the negative of $\vec{a}$ is $-\vec{a}$.

Definition: ***Two vectors are equal,*** **if they are unidirectional and their magnitudes are equal.**

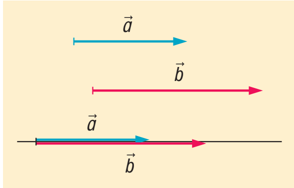


Figure 94

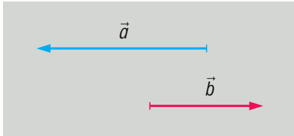


Figure 95

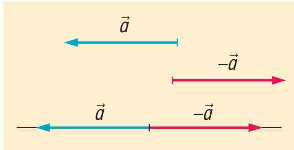


Figure 96

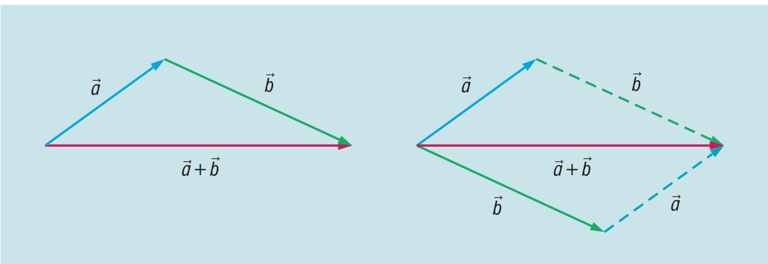
The sum of vectors

We defined the sum of vectors with the help of the parallel translation.

Definition: **The sum of the vectors $\vec{a}$ and $\vec{b}$ (notation: $\vec{a}+\vec{b}$) is the vector of the parallel translation that can replace the parallel translations defined by $\vec{a}$ and $\vec{b}$ in succession.**

Graphically we can add two vectors based on the **triangle rule** or the **parallelogram rule.** *(Figure 97)*

Figure 97



The addition of vectors is

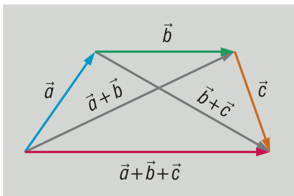
(1) a **commutative** operation, i.e.

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}.$$

(2) an **associative** operation *(Figure 98)*, i.e.

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})=\vec{a}+\vec{b}+\vec{c}.$$

Figure 98



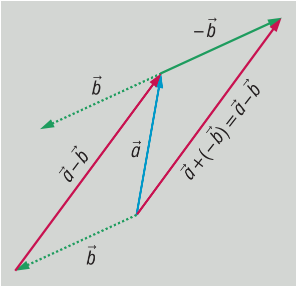


Figure 99

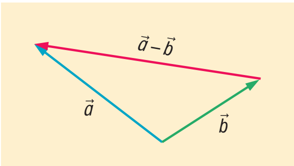


Figure 100

The difference of vectors

DEFINITION: The *difference of the vectors* $\vec{a}$ and $\vec{b}$ (notation: $\vec{a}-\vec{b}$) is the vector $\vec{a}+(-\vec{b})$. *(Figure 99)*

Graphically the difference of two vectors can be constructed as shown in figure 100.

Based on the previous definitions the graphic result of the operation $\vec{a}+(-\vec{a})=\vec{a}-\vec{a}$ is a point, i.e. a vector (directed line segment) the starting point and the end-point of which coincide.

DEFINITION: The vector, the magnitude of which is 0, *nullvek-* is called the *zero vector.* (Notation: $\vec{0}$ or 0.)

According to the conventions the direction of $\vec{0}$ is arbitrary, i.e. it is unidirectional with any vector.

Scalar multiplication (multiplying a vector by a number)

DEFINITION:

1. If $\vec{a}\neq\vec{0}$ and α are arbitrary real numbers, then $\alpha\cdot\vec{a}$ is a vector the magnitude of which is $|\alpha|\cdot|\vec{a}|$ and in the case of $\alpha>0$ it is unidirectional with $\vec{a}$, in the case of $\alpha<0$ it is in the opposite direction as $\vec{a}$.

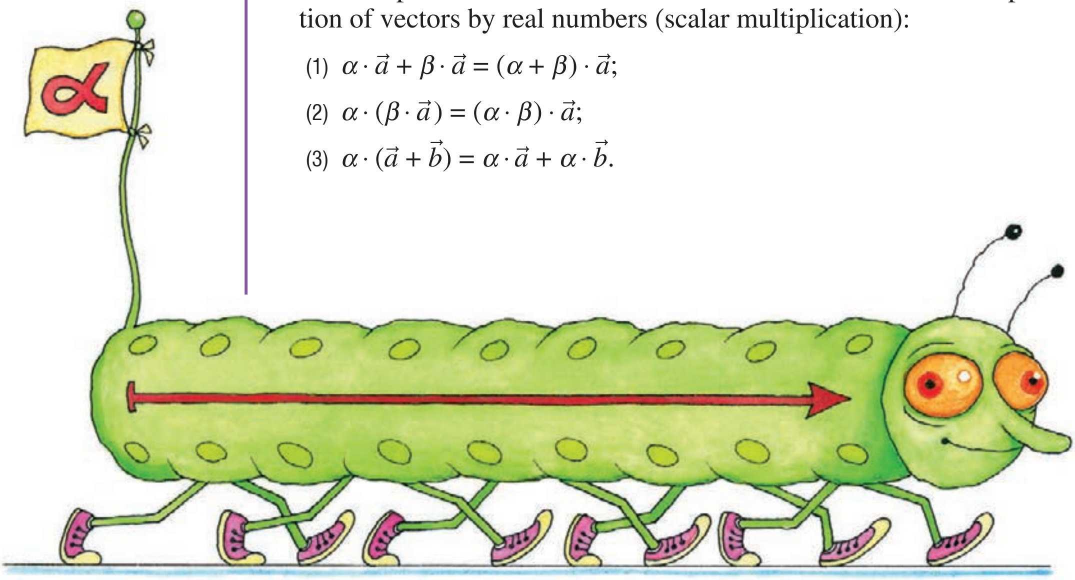
2. If $\vec{a}=\vec{0}$, then $\alpha\cdot\vec{a}=\vec{0}$ for any real number α.

It can be proven that the identities below are satisfied for the multiplication of vectors by real numbers (scalar multiplication):

(1) $\alpha\cdot\vec{a}+\beta\cdot\vec{a}=(\alpha+\beta)\cdot\vec{a}$;

(2) $\alpha\cdot(\beta\cdot\vec{a})=(\alpha\cdot\beta)\cdot\vec{a}$;

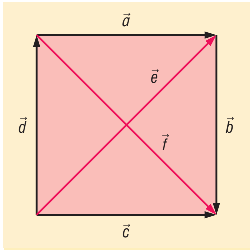
(3) $\alpha\cdot(\vec{a}+\vec{b})=\alpha\cdot\vec{a}+\alpha\cdot\vec{b}$.



Exercises

1. Six vectors defined by the four vertices of a square can be seen in the figure. Choose the ones that are

   a) equal; *b)* parallel; *c)* negatives.



2. Express the following vectors by taking the vectors of the previous exercise as a basis.

   a) $\vec{a}+\vec{b}$; *b)* $\vec{c}-\vec{d}$; *c)* $\vec{b}+\vec{d}$;

   d) $\vec{a}+\vec{b}+\vec{d}$; *e)* $\vec{b}+\vec{e}$; *f)* $\vec{d}-\vec{e}$;

   g) $\vec{c}+\vec{d}$; *h)* $\vec{e}+\vec{f}$; *i)* $(\vec{e}-\vec{c})+\vec{a}$; *j)* $(\vec{e}-\vec{f})+(\vec{a}-\vec{b})$.

3. Take $\vec{a}$ different from $\vec{0}$, and construct the following vectors.

   a) $2\cdot\vec{a}$; *b)* $-3\cdot\vec{a}$; *c)* $-\frac{3}{4}\cdot\vec{a}$; *d)* $\sqrt{2}\cdot\vec{a}$;

   e) $\frac{1}{2}\cdot(3\cdot\vec{a})$; *f)* $-\sqrt{2}\cdot(\sqrt{8}\cdot\vec{a})$.

4. Take the non-parallel vectors $\vec{a}$ and $\vec{b}$ different from the zero vector. Construct the following vectors.

   a) $\vec{a}+\vec{b}$; *b)* $\vec{a}-\vec{b}$; *c)* $2\cdot\vec{a}+\vec{b}$; *d)* $\vec{a}-\frac{\vec{b}}{2}$;

   e) $-(\vec{a}+\vec{b})$; *f)* $2\cdot\left(\vec{b}-\frac{\vec{a}}{3}\right)$.

5. Give a simpler form of the following vectors.

   a) $2\cdot(\vec{a}+\vec{b})+(3\cdot\vec{a}-\vec{b})$; *b)* $\frac{1}{2}\cdot(\vec{b}-2\cdot\vec{a})-3\cdot\left(\vec{a}+\frac{\vec{b}}{3}\right)$;

   c) $-4\cdot(\vec{b}+3\cdot\vec{a})-\sqrt{2}\cdot(-\vec{a}-4\cdot\vec{b})$.

6. Prove that if it is satisfied for the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ that $\frac{\vec{a}+\vec{c}}{\sqrt{3}}=\frac{\sqrt{3}}{3}\cdot\vec{b}+\frac{\sqrt{3}}{3}\cdot\vec{d}$, then $\vec{b}-\vec{a}=\vec{c}-\vec{d}$.

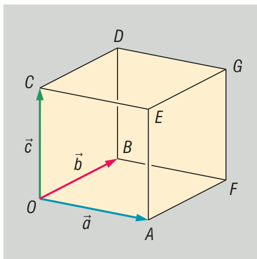
7. The edge vectors starting from the vertex O of the cube shown in the figure are $\vec{a}$, $\vec{b}$, $\vec{c}$.

   Express the following vectors with the help of these three vectors.

   a) $\overrightarrow{OF}$; *b)* $\overrightarrow{OD}$; *c)* $\overrightarrow{OG}$;

   d) $\overrightarrow{GB}$; *e)* $\overrightarrow{DE}$; *f)* $\overrightarrow{FD}$;

   g) $\overrightarrow{DA}$; *h)* $\overrightarrow{CF}$.

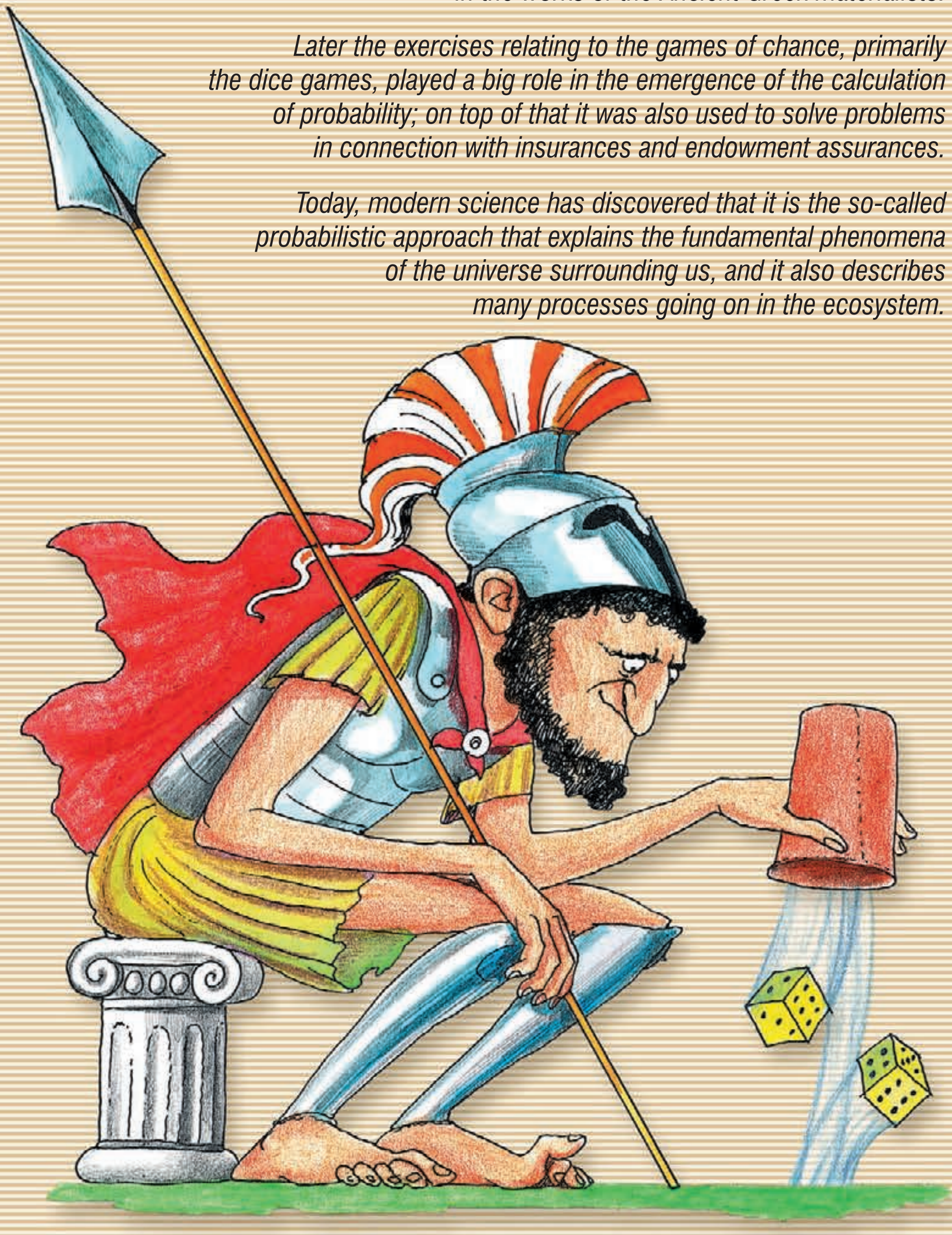


Calculation of probability

The concept of probability already played an important role back in the philosophy of Ancient Greece. The idea that principles experienced in nature emerge through the mass of randomness first arose in the works of the Ancient Greek materialists.

Later the exercises relating to the games of chance, primarily the dice games, played a big role in the emergence of the calculation of probability; on top of that it was also used to solve problems in connection with insurances and endowment assurances.

Today, modern science has discovered that it is the so-called probabilistic approach that explains the fundamental phenomena of the universe surrounding us, and it also describes many processes going on in the ecosystem.



3. Experiments, frequency, relative frequency, probability

Suppose we have an experiment where we roll a die and we observe the number rolled. Let the event A be that the result of the roll is 6. We repeated the experiment 100 times and we counted that the event A occurred 15 times, i.e. the frequency of the event A is 15. The event A occurred in $\frac{15}{100} = \frac{3}{20}$ of the number of the experiments, i.e. the *relative frequency* of the event A is $\frac{15}{100}$.

frequency, relative frequency

DEFINITION: If out of n experiments the event A occurred k times ($k \le n$), then k is called the *frequency* of the event A, and $\frac{k}{n}$ is called the *relative frequency* of the event A.

Example 1

Let us roll a die 100 times. Let us give the frequency and the relative frequency of the following events.

A: The result of the roll is an even prime.

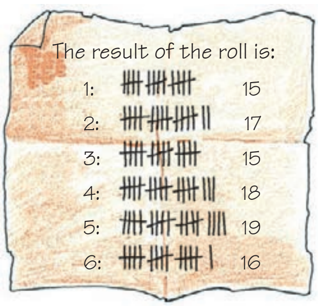
B: The result of the roll is an odd prime.

C: The number rolled is a prime.

D: The number rolled is at most 6.

Solution

The frequency and the relative frequency of the corresponding events based on one sequence of rolling:



$A = \{2\}$, the frequency of the event is: 17, and its relative frequency is: $\frac{17}{100}$.

$B = \{3, 5\}$, the frequency of the event is: 34, its relative frequency is: $\frac{34}{100}$.

$C = \{2, 3, 5\}$, the frequency of the event is: 51, its relative frequency is: $\frac{51}{100}$.

$D = \{1, 2, 3, 4, 5, 6\}$, the frequency of the certain event is 100, its relative frequency is: 1.

Let us observe that the events A and B are mutually exclusive: $A \cap B = \varnothing$.

The frequency of the event $A \cup B = C$ is the sum of the frequency of the events A and B, the relative frequency of the union $A \cup B$ is the sum of the relative frequency of the terms.

- **The relative frequency cannot be negative: since $0 \le k \le n$ and $n > 0$, therefore $0 \le \frac{k}{n} \le 1$.**
- **The relative frequency of the certain event is 1, because the certain event occurs in every experiment, thus $k = n$.**
- **The relative frequency of the union of mutually exclusive events is the sum of the relative frequency of the terms.**

By conducting many similar sequences of experiments we can observe that the relative frequency of a given event deviates around a number. The more experiments we conduct, the less the deviation is in general.

The *probability* of the event A is considered to be the number around which the relative frequency is deviating.

The notation for the probability of the event A is: $P(A)$.

Note: We do not define the probability but we determine it with axioms, which are resulting corresponding to the experience based on the properties of the relative frequency.

Axiom I: If A is an arbitrary event, then $P(A) \ge 0$.

Axiom II: $P(H) = 1$.

Axiom III: If A and B are arbitrary events for which $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Example 2

The problem of the three cubes: The Duke of Tuscany wrote a letter to Galileo Galilei in which he brought up the following problem and he was waiting for the answer of the wise man. At that time the people often played dice games: they rolled three dice at once and observed the sum of the numbers rolled on the three dice. They counted that the sum can be 9 in six ways, and it can be 10 also in six ways, hence:

$$9 = 1+2+6 = 1+3+5 = 1+4+4 = 2+2+5 = 2+3+4 = 3+3+3$$
$$10 = 1+3+6 = 1+4+5 = 2+2+6 = 2+3+5 = 2+4+4 = 3+3+4.$$

Yet during the games they realised that the sum is 10 more often than 9. Why does the counting contradict the experience?

Solution

Let us colour the dice red, blue and green. We considered the cases when we rolled 1, 2 and 6 with the three dice as one case. Since the dice are colourful it can be seen that it makes a difference whether the 1 is rolled with the red die or with the blue die. Let us count the number of cases while taking the colours into account. Three different numbers can be rolled in $3 \cdot 2 \cdot 1$ different ways, because 3 numbers can be rolled with the red die, then 2 numbers can be rolled with the blue die, and finally 1 number can be rolled with the green die. If two numbers

GALILEO GALILEI (1564–1642): the work with the title *Considerazione sopra il Giuoco dei Dadi* (Thoughts about Dice Games) was found in his legacy, in which he dealt with such problems.

probability

ANDREJ NYIKOLAJEVICS KOLMOGOROV (1903–1987), Russian mathematician was the modern mathematical founder of the calculation of probability. His work with the title *Basics of Probability Theory* was first published in 1933 in German, then in 1936 in Russian. In this work he wrote down the axioms relating to the probability.

Sum of 9	
1+2+6	6 elementary events
1+3+5	6 elementary events
1+4+4	3 elementary events
2+2+5	3 elementary events
2+3+4	6 elementary events
3+3+3	1 elementary events
	total: **25** elementary events

Sum of 10	
1+3+6	6 elementary events
1+4+5	6 elementary events
2+2+6	3 elementary events
2+3+5	6 elementary events
2+4+4	3 elementary events
3+3+4	3 elementary events
	total: **27** elementary events

are equal among the numbers rolled, for example 1+ 4 + 4 are rolled, then the 1 can be rolled with any of the three coloured dice, thus it means 3 cases with the coloured dice. If all three numbers rolled are equal, then it obviously counts as 1 case with colourful dice too. Thus by taking the colours into account the sum can be 9 in 25 different ways and it can be 10 in 27 different ways. The experience showed that the people rolled 10 as the sum more times than 9, which confirms that the dice should be considered as different.

In reality two dice are always different even if we cannot distinguish them visibly.

Exercises

1. We conduct 100 experiments with two differently coloured (black, white) dice, and we observe the frequency and the relative frequency of the following events.
   A: The numbers rolled with the two dice are equal.
   B: The number rolled with the black die is greater than the number rolled with the white die.
   C: The number rolled with the white die is greater than the number rolled with the black die.
2. Experiment with two coins. Flip the two coins 100 times in succession and observe the frequency and the relative frequency of the following events: the number of heads on the two coins is 0, 1, 2.
3. We put 4 pieces of paper in a hat with the numbers 1, 2, 3, 4 written on them respectively. The experiment is that we draw a piece of paper from the hat three times in succession at random, and then we write down the number and replace the piece of paper. Thus a three-digit number results. By conducting the experiment 30 times examine the frequency and the relative frequency of the following events.
   A: The three-digit number is even.
   B: The three-digit number is divisible by 3.
   What do you experience if you conduct the experiment 60 times or 90 times?
4. When rolling three different dice if the sum of the numbers rolled is greater than 10, then we win, otherwise we lose. In how many ways can we win?

GAME

The "quizmaster" rolls 7 numbers with a die in succession. Every player immediately writes this number in the place of one of the selected digits in the addend numbers. Whoever has the greatest sum wins the game. What is the greatest possible sum? And what is the probability of occuring?

$$\begin{array}{r} \square\square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$